Flow of viscous incompressible fluid through circular pipe

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Abstract

In this paper, some important types of fluid flows are expressed. Then, the rate of flow and continuity equation of a fluid flow through a pipe are studied. The flow of viscous incompressible fluid through circular pipe is mainly discussed with examples.

1. Introduction

This paper deals with the flow of fluids which is viscous and flowing at very low velocity. At low velocity the fluid moves in layers. Each layer of fluid slides over the adjacent layer. Due to relative velocity between two layers the velocity gradient $\frac{du}{dy}$ exists and hence a shear stress $\tau = \mu \frac{du}{dy}$ acts on the layers.

For the flow of viscous fluid through circular pipe, the velocity distribution across a section, the ratio of maximum to average velocity, the shear stress distribution and drop of pressure for a given length is to be determined. The flow through the circular pipe will be viscous or laminar, if the Reynolds number (Re) is less than 2000. The expression for Reynold number is given by

$$Re = \frac{\rho \bar{u} D}{\mu},$$

where ρ = density of fluid flowing through pipe

$\bar{u}$ = average velocity of fluid

D = diameter of pipe and

μ = viscosity of fluid.

2. Some Important Types of Fluid Flows

Steady and unsteady flows

Steady flow is defined as that type of flow in which the fluid characteristics like velocity V, pressure p, density ρ, etc., at a point do not change with time. Thus for steady flow, mathematically, we have

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$$\left(\frac{\partial V}{\partial t}\right)_{(x_0,y_0,z_0)} = 0\,,\quad \left(\frac{\partial p}{\partial t}\right)_{(x_0,y_0,z_0)} = 0\,,\quad \left(\frac{\partial \rho}{\partial t}\right)_{(x_0,y_0,z_0)} = 0\,,$$

where (x_0, y_0, z_0) is a fixed point in fluid field.

Unsteady flow is that type of flow, in which the velocity, pressure or density at a point changes with respect to time. Thus, mathematically, for unsteady flow

$$\left(\frac{\partial V}{\partial t}\right)_{(x_0,y_0,z_0)} \neq 0\,,\quad \left(\frac{\partial p}{\partial t}\right)_{(x_0,y_0,z_0)} \neq 0\,,\quad \left(\frac{\partial \rho}{\partial t}\right)_{(x_0,y_0,z_0)} \neq 0\,.$$

Uniform and non–uniform flows

Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (i,e., length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t=\text{constant}} = 0$$

where ∂V = change of velocity

∂s = length of flow in the direction s.

Non–uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non–uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t=\text{constant}} \neq 0\,.$$

Laminar and turbulent flows

Laminar flow is defined as that type of flow in which the fluid particles move along well–defined paths or stream lines and all the stream lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream line flow or viscous flow.

Turbulent flow is that type of flow in which the fluid particles move in a zig–zag way. Due to the movement of fluid particles in a zig–zag way, the eddies formation takes place which are responsible for high energy loss.

For a pipe flow, the type of flow is determined by a non–dimensional number $\frac{\rho \bar{u} D}{\mu}$ called the Reynold number.

If the Reynold number is less than 2000, the flow is called laminar. If the Reynold number is more than 4000, it is called turbulent flow. If the Reynold number lies between 2000 and 4000, the flow may be laminar or turbulent.

Compressible and incompressible flows

Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density ρ is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq \text{constant.}$$

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = \text{constant.}$$

Rotational and irrotational flows

Rotational flow is that type of flow in which the fluid particles while flowing along stream lines, also rotate about their own axis.

If the fluid particles while flowing along stream lines, do not rotate about their own axis then that type of flow is called **irrotational flow**.

3. Rate of Flow or Discharge Q

It is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel.

For an incompressible fluid (or liquid), the rate of flow or discharge is expressed as the volume of fluid flowing across the section per second.

For compressible fluids, the rate of flow is usually expressed as the weight of fluid across the section.

Thus

(i) for liquids, the units of Q are m^3/s or litre /s

(ii) for gases, the units of Q is Newton /s.

We consider a liquid flowing through a pipe in which

A = cross–sectional area of pipe

$\bar{u}$ = average velocity of fluid across the section.

Then, the discharge is

$$Q = A\bar{u}. \tag{1}$$

4. Continuity Equation

The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross–section, the quantity of fluid per second is constant.

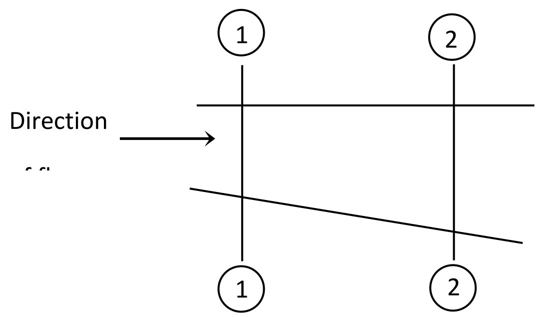

Figure 1. Fluid flowing through a pipe

We consider two cross–sections of a pipe as shown in Figure 1.

Let $\bar{u}_1$ = average velocity at across–section 1–1,

ρ_1 = density at section 1–1,

A_1 = area of pipe at section 1–1

and $\bar{u}_2, \rho_2, A_2$ are corresponding values at section 2–2.

Then rate of flow at section 1–1 is $\rho_1 A_1 \bar{u}_1$ and

The rate of flow at section 2–2 is $\rho_2 A_2 \bar{u}_2$.

According to law of conversation of mass,

the rate of flow at section 1–1 = the rate of flow at section 2–2

$$\rho_1 A_1 \bar{u}_1 = \rho_2 A_2 \bar{u}_2. \tag{2}$$

Equation (2) is applicable to the compressible as well as incompressible fluids and is called **continuity equation**.

If the fluid is incompressible, then $\rho_1 = \rho_2$ and continuity equation (2) reduces to

$$A_1 \bar{u}_1 = A_2 \bar{u}_2. \tag{3}$$

5. Flow of Viscous Fluid Through Circular Pipe

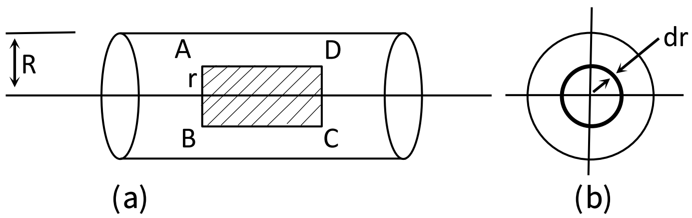

Figure 2. Viscous flow through a pipe

We consider a horizontal pipe of radius R. The viscous fluid flowing from left to right in the pipe as shown in Figure 2 (a).

We consider a fluid element of radius r, sliding in a cylindrical fluid element of radius $(r + dr)$. Let the length of fluid element be Δx.

If p is the intensity of pressure on the face AB, then the intensity of pressure on face CD will be $\left(p + \frac{\partial p}{\partial x}\Delta x\right)$.

Then the forces acting on the fluid element are

(i) the pressure force $p\pi r^2$ on face AB

(ii) the pressure force $\left(p + \frac{\partial p}{\partial x}\Delta x\right)\pi r^2$ on face CD

(iii) the shear force $2\pi r\,\tau\,\Delta x$ on the surface of fluid element.

As there is no acceleration, hence the summation of all forces in the direction of flow must be zero.

$$p\,\pi r^2 - \left(p + \frac{\partial p}{\partial x}\Delta x\right)\pi r^2 - 2\pi r\,\tau\,\Delta x = 0$$

$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2}. \qquad (4)$$

The shear stress τ across a section varies with r as $\frac{\partial p}{\partial x}$ across a section is constant. Hence shear stress distribution across a section is linear as shown in Figure 3(a).

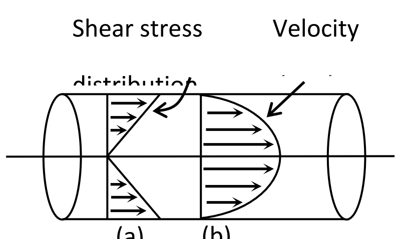

Figure 3. Shear stress and velocity distribution across a section

Velocity distribution

To obtain the velocity distribution across a section, the value of shear stress $\tau = \mu \frac{du}{dy}$ is substituted in equation (4).

But this relation $\tau = \mu \frac{du}{dy}$, y is measured from the pipe wall.

Hence $y = R - r$ and $dy = -dr$.

So, $\tau = -\mu \frac{du}{dr}$.

Substituting this value in equation (4), we get

$$\frac{du}{dr} = \frac{1}{2\mu}\frac{\partial p}{\partial x} r.$$

Integrating this above equation with respect to r, we get

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 + C, \qquad (5)$$

where C is the constant of integration and its value is obtained from the boundary condition that at $r = R$, $u = 0$. We get

$$C = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2.$$

Substituting this value of C in equation (5), we get

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 - \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left(R^2 - r^2\right). \qquad (6)$$

In equation (6), values of μ, $\frac{\partial p}{\partial x}$ and R are constant, which means the velocity u varies with the square of r. Thus equation (6) is the equation of parabola.

This shows that the velocity distribution across the section of a pipe is parabolic. This velocity distribution is shown in Figure 3(b).

Ratio of maximum velocity to average velocity

The velocity is maximum, when $r = 0$ in equation (6). Thus maximum velocity U_{max} is obtained as

$$U_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2. \qquad (7)$$

The average velocity $\bar{u}$ is obtained by dividing the discharge of the fluid across the section by the area of the pipe (πR^2). The discharge Q across the section is obtained by considering the flow through a circular ring element of radius r and thickness dr as shown in Figure 2 (b). The fluid flowing per second through this elementary ring

$$dQ = \text{velocity at a radius r} \times \text{area of ring element}$$
$$= 2u\pi r\,dr$$
$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left(R^2 - r^2\right)2\pi r\,dr.$$

Then,
$$Q = \int_0^R dQ$$
$$= \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)2\pi\int_0^R\left(R^2 - r^2\right)r\,dr$$
$$= \frac{\pi}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^4. \qquad (8)$$

Therefore, average velocity is

$$\bar{u} = \frac{Q}{\text{Area}}$$
$$= \frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right)R^2. \qquad (9)$$

Dividing equation (6) by equation (9),

$$\frac{U_{max}}{\bar{u}} = \frac{-\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2}{\frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^2} = 2.0.$$

So,the ratio of maximum velocity to average velocity is 2.

That is,

$$U_{max} = 2\bar{u}.$$

Drop of pressure for a given length (L) of a pipe

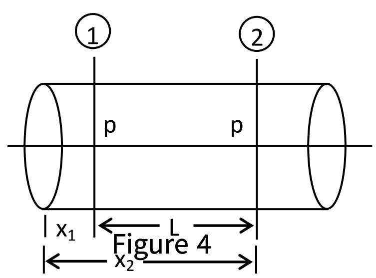

Figure 4

From equation (9), we have

$$\bar{u} = \frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right)R^2$$

$$\left(\frac{-\partial p}{\partial x}\right) = \frac{8\mu\bar{u}}{R^2}.$$

Integrating the above equation with respect to x, we get

$$-\int_{p_2}^{p_1} dp = \int_{x_2}^{x_1} \frac{8\mu\bar{u}}{R^2}dx$$

$$p_1 - p_2 = \frac{8\mu\bar{u}}{R^2}\left[x_2 - x_1\right]$$

$$= \frac{8\mu\bar{u}}{R^2}L$$

$$= \frac{32\mu\bar{u}L}{D^2}, \qquad (10)$$

where $p_1 - p_2$ is the drop of pressure.

Then, the loss of pressure head is $\dfrac{p_1 - p_2}{\rho g}$.

So, $$\frac{p_1 - p_2}{\rho g} = \frac{32\mu\bar{u}L}{\rho g D^2}. \qquad (11)$$

Equation (11) is called **Hagen Poiseuille Formula.**

Example (1)

A crude oil of viscosity 0.97 poise and relative density 0.9 is flowing through a horizontal circular pipe of diameter 100 mm and of length 10m.

Therefore, $\mu = 0.97 \text{ poise} = \dfrac{0.97}{10} = 0.097 Ns/m^2$.

Since the density of water is 1000 kg /m^3,

the density of oil is $\rho = 0.9(1000) = 900 kg/m^3$.

If 100 kg of the oil is collected in a tank in 30 seconds,

we can calculate difference of pressure ($p_1 - p_2$) at the two ends of the pipe.

The difference of pressure ($p_1 - p_2$) for viscous flow is given by

$$p_1 - p_2 = \frac{32\mu\bar{u}L}{D^2},$$

where $\bar{u}$ = average velocity = $\dfrac{Q}{Area}$.

Now, mass of oil /sec $= \dfrac{100}{30} kg/s.$

Since the mass of oil/sec is $\rho Q = 900Q$,

$$\frac{100}{30} = 900Q.$$

Then, $$Q = \frac{100}{30}\frac{1}{900} = 0.0037 m^3/s. \tag{12}$$

The average velocity is

$$\bar{u} = \frac{Q}{Area} = \frac{0.0037}{\frac{\pi}{4}D^2} = \frac{0.0037}{\frac{\pi}{4}(0.1)^2} = 0.471 m/s. \tag{13}$$

For laminar flow, the Reynolds number (Re) is less than 2000. Let us calculate the Reynolds number for this example.

Reynolds number is $Re = \dfrac{\rho \bar{u} D}{\mu}$,

where $\rho = 900$, $\bar{u} = 0.471$, $D = 0.1m$, $\mu = 0.097$.

Then, $$Re = 900\,\frac{0.471\,0(0.1)}{0.097} = 436.91\,. \tag{14}$$

As Reynolds number is less than 2000, the flow is laminar.

$$\begin{aligned} p_1 - p_2 = \frac{32\mu\bar{u}L}{D^2} &= \frac{32(0\,.097\,)(0.471)(10)}{(0.1)^2} N/m^2 \\ &= 1462.28 N/m^2 \\ &= (1462.28)10^{-4} N/cm^2 \\ &= 0.1462 N/cm^2. \end{aligned} \tag{15}$$

Example(2)

A laminar flow is taking place in a pipe of diameter 200 mm. The maximum velocity is 1.5 m /s.

That is, D = 200mm = 0.20 m and U_{max} =1.5m/s.

We can find the mean velocity $\bar{u}$ and the radius r at which this occurs.

Since $U_{max} = 2\bar{u}$, $\bar{u} = \dfrac{1.5}{2.0} = 0.75 m/s.$

The velocity u at any radius r is given

$$u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R^2 - r^2\right] = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2\left[1 - \frac{r^2}{R^2}\right].$$

But from equation U_{max} is given by $U_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2$,

$$u = U_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]. \tag{16}$$

Now, the radius r at which $u = \bar{u} = 0.75\text{m/s}$,

$$0.75 = 1.5\left[1 - \left(\frac{r}{D/2}\right)^2\right]$$

$$\frac{0.75}{1.5} = 1 - \left(\frac{r}{0.1}\right)^2$$

$$\left(\frac{r}{0.1}\right)^2 = 1 - \frac{0.75}{1.5} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\frac{r}{0.1} = \sqrt{\frac{1}{2}} = \sqrt{0.5}$$

$$r = 0.0707\text{m} = 70.7\text{mm}.$$

We can also calculate the velocity at 4 cm from the wall of the pipe.

$$r = R - 4.0 = 10 - 4.0 = 6.0\text{cm} = 0.06\text{ m}.$$

Then, the velocity at a radius 0.06 m $= 1.5\left[1 - \left(\frac{0.06}{0.1}\right)^2\right]$

$$= 1.5\left(1.0 - 0.36\right) = 0.96\,\text{m/s}.$$

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